Theory of hybrid systems. I. The operator formulation of classical mechanics and semiclassical limit

Slobodan Prvanović and Zvonko Marić
Institute of Physics, P.O. Box 57, 11080 Belgrade, Yugoslavia

Abstract

The algebra of polynomials in operators that represent generalized coordinate and momentum and depend on the Planck constant is defined. The Planck constant is treated as the parameter taking values between zero and some nonvanishing h_0. For the second of these two extreme values, introduced operatorial algebra becomes equivalent to the algebra of observables of quantum mechanical system defined in the standard manner by operators in the Hilbert space. For the vanishing Planck constant, the generalized algebra gives the operator formulation of classical mechanics since it is equivalent to the algebra of variables of classical mechanical system defined, as usually, by functions over the phase space. In this way, the semiclassical limit of kinematical part of quantum mechanics is established through the generalized operatorial framework.
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1 Introduction

The hybrid systems are those consisting of two distinguished subsystems. We are interested in situation when one of these subsystems is quantum and the other one classical mechanical, where, of course, by quantum mechanical system we mean such system whose behavior is properly described in terms of

quantum mechanics. Similarly, concepts and relations of classical mechanics offer the complete formal description of the second subsystem. On the other side, we believe that "hybrid system" is not bad terminological solution for a physical system whose classical and quantum variants are simultaneously considered.

Theory of hybrid systems should incorporate classical and quantum mechanics on equal footing, together with transitions from one to the other. The correspondence principle, or the quantization and dequantization procedure, should be the essential characteristic of this theory. In other words, it should offer mathematical framework that supports simultaneous meaningful formulation of both mechanics. Within it, classical and quantum mechanics should appear as the same regarding the structural characteristics. In particular, they should have one and the same algebraic product and one and the same Lie bracket and, if one of them is formulated in terms of differential geometry, then the other should be, too. Moreover, fundamental entities of classical and quantum mechanics should be interrelated in unambiguous and correct way.

Theory of hybrid systems should give the proper description of interaction between quantum and classical systems. By proper we mean that the collapse of state of quantum system, which can happen when this system interacts with the classical one, *e.g.*, in the process of measurement, should be explained/described without going into circular argumentation.

With such theory one certainly would not solve all problems of classical and quantum mechanics. This, after all, is not its purpose. By exploiting the juxtaposition of classical and quantum mechanics, this theory should give better insight in the two mechanics and their relations. Eventually, this will be enough for finding answers to some important questions.

After this rather general statements, let us give an outlook of the program and strategy of our approach to the listed requests. In the first paper of a series addressing hybrid systems we shall propose new framework that offers simultaneous representations of classical and quantum mechanics instead of using the framework of functions over the phase space, see Argawal and Wolf (1970); Guz (1984); Jordan and Sudarshan (1961); Mehta (1964); Misra and Shankara (1968); Riccia and Wiener (1966); Shirokov (1976) and Uhlhorn (1956). More concretely, we shall define the general algebra of operators that depend on the Planck constant. We shall take this constant as the parameter which takes value between some nonvanishing h_0 and zero.

For h_0 the algebra will reduce in one which is isomorphic to the algebra of quantum mechanical system defined in standard fashion. The other extreme value will be attributed to the classical mechanics. That is, for the vanishing Planck constant, the generalized operator algebra will reduce to the algebra that is in one-to-one correspondence with the well-known algebra of variables of classical mechanical system. Looked from another angle, this will mean that there is an operatorial representation of classical mechanics, as well as of quantum mechanics, and that semiclassical limit (dequantization) is established on the kinematical level through the general algebra. The dynamics will be just mentioned in this paper since it asks for longer exposition.

The Lie algebraical structure of classical and quantum mechanics is important for physics, see Emch (1972). The Lie bracket, or product, in quantum mechanics is closely connected to the ordering rule. In order to complete the correspondence principle, *i.e.*, to include quantization, one has to analyze this topics as well. This we shall do in the second article of this series by defining the ordering rule - symmetrized product, of quantum mechanics. This product will be the one according to which the algebras in both classical and quantum mechanics should be formed. In the classical mechanics, where one deals with the commutative algebra, one can apply without problems any ordering rule, so one can use symmetrized product as the algebraical one. Without going into details, let us mention that by using the symmetrized product we are going to propose, one can define the symmetrized Poisson bracket which can substitute the commutator in quantum mechanics becoming its Lie bracket. This bracket can be used in classical mechanics as well (the explanation is as for the algebraic product). Having one and the same algebraical product and one and the same Lie bracket for both mechanics, we shall be able to propose, we believe, unambiguous - obstruction free, quantization prescription. (The ordering rule and obstructions to quantization were previously discussed in Arens and Babit (1965); Chernoff (1995); Cohen (1966); Gotay (1980) and (1996) and Kerner and Sutcliffe (1970).) Moreover, the dynamical equations of both classical and quantum mechanics will appear to be one and the same - the operatorial version of the Liouville equation.

In the third article we shall unify the results of the first two. Namely, we shall start with the general operators of coordinate and momentum which depend on the Planck constant, then we shall introduce one algebraical product and one Lie algebraical product and, after considering the above mentioned

extreme values of Planck constant, we shall establish the semiclassical limit of quantum mechanics for complete algebraical and Lie algebraical settings. Then, because observables and variables are functions of coordinate and momentum and since the functions are 0-forms in differential geometry, we will go one step further in the exploration of common characteristics of classical and quantum mechanics, *i.e.*, we will find other basic entities of differential geometry in case of quantum mechanics, see Strocchi (1966); Heslot (1985) and Ashtekar and Schiling (1997) for related considerations. We will define n-forms, the Hamiltonian vector fields and symplectic structure of quantum mechanics and these in such operatorial forms that they become equivalent to the corresponding ones of classical mechanics in the semiclassical limit. This will lead to the conclusion that, regarding all aspects, classical and quantum mechanics are just two cases of one general theory. All structural characteristics of these mechanics will appear to be the same while the crucial difference will be in operators representing coordinate and momentum.

In the third article we shall propose the symmetrized Poisson bracket for the case of more than one degree of freedom. This generalization is not so trivial problem, as it might look like on the first sight, since it should be related to the dynamical equation for the hybrid systems - classical and quantum systems in interaction, see Prvanović and Marić (2000) and references therein.

Regarding the interaction between classical and quantum systems, it should be said that one concrete example - the process of measurement, we have considered in quoted article. The results of this paper were that the dynamical equation of hybrid systems can produce noncausal evolution and that the state of quantum system collapses because of the non-negativity of probability. For finding this the operator formulation of classical mechanics was extremely useful. Moreover, this formulation was necessary for the analysis of subsequent problem: if the equation of motion can produce noncausal evolution, then what within this equation one can find as responsible for some nonlinearity which will allowe one to design a formal model of the collapse. Due to these, the quoted article is strongly connected (or belongs) to the series of papers concerned with the theory of hybrid systems being oriented to its application.

2 Basic definitions

The present considerations will address a part of kinematical aspect of quantum and classical mechanics in order to investigate the semiclassical limit of quantum mechanics, see Shirokov (1976); Werner (1995) and references therein. For this intention, it is needed to use the same mathematical framework for both classical and quantum mechanics. In contrast to the usual approach, where functions over the phase space were used for representations of both mechanics and discussions of their relationship, our proposal is based on the operatorial representations. That is, instead of transforming the standard operatorial formulation of quantum mechanics into the formulation comparable to the standard one of classical mechanics, we expose the most important features of classical mechanics in the form of operators acting in the same space where quantum representatives do. The operator formulation of classical mechanics used here is similar to those given in Sherry and Sudarshan (1978) and (1979) and Gautam *et al.* (1979). Some connections between our proposal and those given in Cohn (1980) and (1983); Muga and Snider (1992) and Sala and Muga (1994) can be found, too.

Our intention is to find such formulations of classical and quantum mechanics in which the main characteristics of these theories are preserved. Then, for the intended formulations the following should hold: 1.) the observables and states are in the 1-1 correspondence with the adequate ones of the standard formulations, 2.) the commutation relations among observables and the relations among the eigenstates of observables are the same as are the corresponding ones of the standard formulations and 3.) the mean values are equal to the corresponding mean values calculated in standard fashions. The mathematical framework that will be used can be seen as a direct product of coordinate and momentum representations of quantum mechanics. In this way, it will mimic the phase space of classical mechanics in the most trivial case.

Let us, firstly, define the generalized h-dependent operator algebra as the algebra of polynomials with real coefficients in operators $\tilde{q}$, $\tilde{p}$ and $\tilde{I}$, which are:

$$\tilde{q} = \hat{q} \otimes \hat{I} \otimes \left[\hat{R}_q + \left(1 - \frac{h}{h_o} \right) \hat{R}_p \right] + \hat{I} \otimes \hat{q} \otimes \hat{R}_p, \tag{1}$$

$$\tilde{p} = \hat{p} \otimes \hat{I} \otimes \hat{R}_q + \hat{I} \otimes \hat{p} \otimes \left[\left(1 - \frac{h}{h_o} \right) \hat{R}_q + \hat{R}_p \right], \quad (2)$$

and $\tilde{I} = \hat{I} \otimes \hat{I} \otimes \hat{I}$. These operators act in $\mathcal{H}_q \otimes \mathcal{H}_p \otimes \mathcal{H}_r$ with q, p and r being here just the indices of these spaces. The first two spaces are the rigged Hilbert spaces and the third is a two-dimensional Hilbert space. More concretely, $\mathcal{H}_q$ and $\mathcal{H}_p$ are formally identical to the rigged Hilbert space of states which is used in the nonrelativistic quantum mechanics for a single system with one degree of freedom when the spin is neglected. The indices q and p serve only to denote that the choice of a basis in these spaces is *a priori* fixed when the semiclassical limit is under considerations. For the basis in $\mathcal{H}_q \otimes \mathcal{H}_p$ we take $|q\rangle \otimes |p\rangle$. Here, $|q\rangle$ and $|p\rangle$ are the eigenvectors of $\hat{q}$ and $\hat{p}$, respectively. Then, $\mathcal{H}_q \otimes \mathcal{H}_p$ can be seen as an imitation of the phase space. The third space is introduced only for the formal reasons and need not to be related to the space of states of the inner degrees of freedom.

The parameter h takes values from 0 to h_o, $0 \leq h \leq h_o$, where h_o is attributed to quantum mechanics - the nonvanishing Planck constant, while, for $h = 0$, the above algebra will be related to the classical mechanics.

The operators $\hat{q}$ and $\hat{p}$ are as the operators representing coordinate and momentum in the standard quantum mechanics. Instead of reviewing their properties, which can be found in all textbooks of quantum mechanics, we only mention that they do not commute: $[\hat{q}, \hat{p}] = i\hbar\hat{I}$ and that they are the Hermitian (self-adjoint). For the projectors $\hat{R}_q$ and $\hat{R}_p$, the following holds:

$$\hat{R}_q \cdot \hat{R}_p = 0, \quad \hat{R}_q \cdot \hat{R}_q = \hat{R}_q, \quad \hat{R}_p \cdot \hat{R}_p = \hat{R}_p,$$

$$\hat{R}_q^\dagger = \hat{R}_q, \quad \hat{R}_p^\dagger = \hat{R}_p, \quad \hat{R}_q + \hat{R}_p = \hat{I},$$

$$\hat{R}_q = |r_q\rangle\langle r_q|, \quad \hat{R}_p = |r_p\rangle\langle r_p|.$$

They, being introduced to ensure desired properties of the polynomials in $\tilde{q}$ and $\tilde{p}$ for the extreme values of h, need not to have physical meaning.

3 Quantum mechanics

When the above algebra of operators is represented with respect to the basis $|q\rangle \otimes |p\rangle \otimes |r_i\rangle$, where $i \in \{q, p\}$ and $|r_i\rangle$ is the eigenvector of $\hat{R}_i$ for the

eigenvalue 1, ($\langle r_i|r_j\rangle = \delta_{i,j}$), then, for $h = h_o$, it becomes equivalent to the (same representation of) algebra formed over:

$$\hat{q}_{qm} = \hat{q} \otimes \hat{I} \otimes \hat{R}_q + \hat{I} \otimes \hat{q} \otimes \hat{R}_p, \tag{3}$$

$$\hat{p}_{qm} = \hat{p} \otimes \hat{I} \otimes \hat{R}_q + \hat{I} \otimes \hat{p} \otimes \hat{R}_p. \tag{4}$$

(We are not going to write these representations explicitly just for the sake of simplicity of expressions. But, this can be easily done knowing that $\langle q|\hat{q}|q'\rangle = q\delta(q-q')$, $\langle q|\hat{p}|q'\rangle = -i\hbar\frac{\partial\delta(q-q')}{\partial q}$, $\langle p|\hat{p}|p'\rangle = p\delta(p-p')$ and $\langle p|\hat{q}|p'\rangle = i\hbar\frac{\partial\delta(p-p')}{\partial p}$.)

The algebra of polynomials with real coefficients in $\hat{q}_{qm}$ and $\hat{p}_{qm}$ and the appropriate eigenvectors are in the one-to-one correspondence with the algebra of observables and eigenstates defined in the standard manner in a single rigged Hilbert space. For example, it holds: $[\hat{q}_{qm}, \hat{p}_{qm}] = i\hbar\tilde{I}$, as it is necessary. On the other hand, due to the fact that $\hat{R}_q$ and $\hat{R}_p$ are idempotent and mutually orthogonal, the standard representation of an observable, *e.g.*, $H(\hat{q}, \hat{p})$, is now translated to:

$$H(\hat{q}_{qm}, \hat{p}_{qm}) = H(\hat{q}, \hat{p}) \otimes \hat{I} \otimes \hat{R}_q + \hat{I} \otimes H(\hat{q}, \hat{p}) \otimes \hat{R}_p.$$

It could be said that this formulation of quantum mechanics simply doubles the standard one. Or, it takes the coordinate representation, multiply it directly with one projector and adds this to the momentum representation, which was directly multiplied with the other projector.

If $|\Psi_i\rangle$ was the eigenstate of $H(\hat{q}, \hat{p})$ for the eigenvalue E_i, then:

$$|\tilde{\Psi}_i\rangle = c_q|\Psi_i\rangle \otimes |a\rangle \otimes |r_q\rangle + c_p|b\rangle \otimes |\Psi_i\rangle \otimes |r_p\rangle,$$

is the eigenstate of $H(\hat{q}_{qm}, \hat{p}_{qm})$ for the same eigenvalue if the coefficients c_q and c_p satisfy the condition $|c_q|^2 + |c_p|^2 = 1$ and if the vectors $|a\rangle$ and $|b\rangle$, that are fixed at the beginning of all considerations being arbitrarily picked, are normalized to one ($\langle a|a\rangle = \langle b|b\rangle = 1$). Moreover, it could be easily checked that all relations among eigenstates ,*a la* $|\tilde{\Psi}_i\rangle$, of the same or different observables are as they were for the corresponding ones, *i.e.*, $|\Psi_i\rangle$, of the standard formulation.

Since our intention is to discuss the semiclassical limit of quantum mechanics after the introduction of new operatorial representations of both classical and quantum mechanics, it is necessary to take $\mathcal{H}_q \otimes \mathcal{H}_p \otimes \mathcal{H}_r$. Of course,

this space is much bigger than $\mathcal{H}$ where quantum mechanical coordinate and momentum are irreducibly represented. Only a subspace of $\mathcal{H}_q \otimes \mathcal{H}_p \otimes \mathcal{H}_r$ that is formed over the basis $|\tilde{\Psi}_i\rangle$ is interpretable for quantum mechanics. It depends on the choice of $|a\rangle$, $|b\rangle$, c_q and c_p which, after being initially fixed, give desired irreducible representation. (As will become obvious, the states of classical system are embedded in the set of operators that act in $\mathcal{H}_q \otimes \mathcal{H}_p \otimes \mathcal{H}_r$.) In other words, having formal reasons, we have purposely skipped over the requirement of irreducible representation of quantum mechanical coordinate and momentum. That this is reducible representation one can check by finding one nontrivial operator, *e.g.*, $\hat{q} \otimes \hat{I} \otimes \hat{R}_p$, which commutes with $\hat{q}_{qm}$ and $\hat{p}_{qm}$. But, one can recover the irreducibility in a formal way and/or extract from $\mathcal{H}_q \otimes \mathcal{H}_p \otimes \mathcal{H}_r$ the physically meaningful part. This is very similar to the approach developed in Streater (1966) and references therein where the request of irreducibility is treated in more relaxed form than usually it is the case.

4 Classical mechanics

In the $|q\rangle \otimes |p\rangle \otimes |r_i\rangle$ representation, but for $h = 0$, the general algebra becomes equivalent to the (same representation of) algebra formed over:

$$\hat{q}_{cm} = \hat{q} \otimes \hat{I} \otimes \hat{I}, \tag{5}$$

$$\hat{p}_{cm} = \hat{I} \otimes \hat{p} \otimes \hat{I}. \tag{6}$$

The algebra of polynomials in $\hat{q}_{cm}$ and $\hat{p}_{cm}$ with real coefficients and the appropriate eigenvectors are in the 1-1 correspondence with the standard formulation of classical mechanics (defined in the framework of functions over phase space). Namely, this algebra is manifestly a commutative one. To the c-number formulation of a classical variable, *e.g.*, $H(q,p)$, now corresponds:

$$H(\hat{q}_{cm}, \hat{p}_{cm}) = H(\hat{q} \otimes \hat{I}, \hat{I} \otimes \hat{p}) \otimes \hat{I}.$$

The vectors $|q_o\rangle \otimes |p_o\rangle \otimes (c_q|r_q\rangle + c_p|r_p\rangle)$ are eigenstates of all classical observables with the real eigenvalues, *e.g.*, $H(q_o, p_o)$ for the above given observable. These vectors now play the role of points in the phase space. The pure states $|q_o\rangle \otimes |p_o\rangle$, taken in dyadic form, can be expressed via $\hat{q}$ and $\hat{p}$ in

the following manner:

$$|q_o\rangle\langle q_o|\otimes|p_o\rangle\langle p_o| = \int\int \delta(q-q_o)\delta(p-p_o)|q\rangle\langle q|\otimes|p\rangle\langle p|dqdp =$$

$$= \delta(\hat{q}-q_o)\otimes\delta(\hat{p}-p_o).$$

Then, being guided by this, the classical (noncoherently) mixed states one can define as:

$$\rho(\hat{q}\otimes\hat{I},\hat{I}\otimes\hat{p})\otimes(c_q|r_q\rangle + c_p|r_p\rangle)(c_q^*\langle r_q| + c_p^*\langle r_p|).$$

All classical states will be the Hermitian, non-negative operators and normalized to $\delta(0)\cdot\delta(0)$ if for $\rho(q,p)$ it holds that: $\rho(q,p)\in\mathbf{R}$, $\rho(q,p)\geq 0$ and $\int\int\rho(q,p)dqdp = 1$, as it is in the phase space formulation of classical mechanics, where $\rho(q,p)$ appears in the coordinate-momentum representation of $\rho(\hat{q}\otimes\hat{I},\hat{I}\otimes\hat{p})$.

One does not need to proceed with translation of other features of the classical mechanics into this framework. Having basic entities, one can do that straightforwardly.

The mean value of the observable, say $\hat{A}$, when the system (classical or quantum) is in the state $\hat{\rho}$, should be calculated within the theory of hybrid systems by the Ansatz:

$$\langle\hat{A}\rangle = \frac{\mathrm{Tr}(\hat{\rho}\hat{A})}{\mathrm{Tr}\hat{\rho}}. \qquad (7)$$

The norm $\delta(0)\cdot\delta(0)$ of classical states will be regularized in this way and the mean values will be equal to those found within the standard formulations for the appropriate variables, observables and states.

The above, and the fact that the phase space formulation of classical mechanics appears through the kernels of the operator formulation in the $|q\rangle\otimes|p\rangle\otimes|r_i\rangle$ representation, can be used as the proof of equivalence of these two formulations. There will be a complete correspondence between the c-number and this operatorial formulation if the dynamical equation is defined as the operatorial version of Liouville equation, where the partial derivations within the Poisson bracket are with respect to the operators $\hat{q}_{cm}$ and $\hat{p}_{cm}$. For now, let the dynamical equation for the above formulation of quantum mechanics be the Schrödinger (von Neumann) equation, as it is in the standard formulation. However, in the next paper we shall show that for

quantum mechanical observables and statistical operators the commutator can be substituted by the symmetrized Poisson bracket (see next section). This will implicate that the dynamical equation of quantum mechanics can be the operatorial version of Liouville equation, as well.

5 Concluding remarks

Since these papers are devoted to the formalism and not to the phenomenology, we are not going in thorough discussions regarding the meaning of limit $h \to 0$, where h is thought to be the natural constant. Instead, let us illustrate what we mean by the semiclassical limit. The Hamilton function $H(q,p)$ of classical mechanics and the Hamiltonian $H(\hat{q},\hat{p})$ of quantum mechanics are represented here by $H(\hat{q}_{cm},\hat{p}_{cm})$ and $H(\hat{q}_{qm},\hat{p}_{qm})$, respectively. If they are addressing the same physical system, say the harmonic oscillator, the last two operators follow from $H(\tilde{q},\tilde{p})$ for $h=0$ and $h=h_o$, respectively. Of course, it is understood that one should work in $|q\rangle \otimes |p\rangle \otimes |r_i\rangle$ representation which, as we have mentioned, we have not proceeded here only for the sake of simplicity of expressions. This means that the semiclassical limit of quantum mechanics is established through the generalized operator algebra since, for the one extreme value of h, it expresses properties characteristic for the quantum mechanics and, for the other extreme value of h, it has classical mechanical ones.

The semiclassical limit holds for each polynomial with real coefficients in $\tilde{q}$ and $\tilde{p}$ no matter how these operators are ordered. This allowed us not to specify explicitly what is the algebraic product of quantum mechanics. The ordering rule, or the symmetrized product, being unrelated to the semiclassical limit, we shall discuss in details in the second paper of this series. In that article, we shall propose new way of looking on the Lie bracket of quantum mechanics, too. Here, let us just mention that the symmetrized product of $\hat{q}^n$ and $\hat{p}^m$, denoted by $\circ$, will be the sum of all different combinations of involved operators divided by the number of these combinations:

$$\hat{q}^n \circ \hat{p}^m = \frac{n!m!}{(n+m)!}(\hat{q}^n\hat{p}^m + \cdots + \hat{p}^m\hat{q}^n),$$

where, in the parenthesis, there should be all different combinations of $n+m$

operators. The product of two monomials, let say $\hat{q}^a \circ \hat{p}^b$ and $\hat{q}^c \circ \hat{p}^d$, will be:

$$\frac{(a+c)!(b+d)!}{(a+b+c+d)!}(\hat{q}^{a+c}\hat{p}^{b+d} + \cdots + \hat{p}^{b+d}\hat{q}^{a+c}).$$

Then, the Lie bracket of quantum mechanics will be:

$$\{f(\hat{q},\hat{p}), g(\hat{q},\hat{p})\}_{\mathbf{S}} = \frac{\partial f(\hat{q},\hat{p})}{\partial \hat{q}} \circ \frac{\partial g(\hat{q},\hat{p})}{\partial \hat{p}} - \frac{\partial g(\hat{q},\hat{p})}{\partial \hat{q}} \circ \frac{\partial f(\hat{q},\hat{p})}{\partial \hat{p}},$$

where, within the polynomials $f(\hat{q},\hat{p})$ and $g(\hat{q},\hat{p})$, the operators of coordinate and momentum are multiplied according to the symmetrized product $\circ$.

Only after having all these results, we will be able to address in complete the problem of dequantization: the semiclassical limit of the kinematical aspect of quantum mechanics and reexpressed dynamical equation of quantum mechanics will be unified in a way leading to the proposition of the semiclassical limit of all quantum mechanical differential geometric entities. Moreover, having defined the symmetrized product and Poisson bracket, we will be able to propose the quantization procedure in the most trivial case. In this way, both directions of the correspondence principle will be covered. The states are seen as the secondary in our proposal. The meaningful states are solutions of the appropriate eigenvalue problems and, because quantum and classical observables are essentially different, they differ, too.

If one is not interested in the semiclassical limit, then the operator formulation of classical mechanics can be simplified. Namely, after noticing that the algebra of variables of classical system with one degree of freedom is isomorphic to the maximal Abelian subalgebra of operators related to the quantum system with two degrees of freedom, one can take $\hat{q} \otimes \hat{I}$, $\hat{I} \otimes \hat{p}$ and $\hat{I} \otimes \hat{I}$ as the basic elements of the operatorial formulation of classical mechanical algebra. This formulation of classical mechanics can find its applicability in, for example, the analysis of the problem of measurement.

Similar proposition one can find in Sherry and Sudarshan (1978) and (1979) and Gautam *et al.* (1979). Without repeating the detailed analysis of these articles, let us mention that a regulating procedure which recovers the classicality was found as necessary there and, for this purpose, the so called principle of integrity was introduced. This is related to the following proposition of our approach: only the algebra formed over $\tilde{q}$, $\tilde{p}$ and $\tilde{I}$ should be considered since only to its elements the physical meaning can be

attributed and this only for the extreme values of Planck constant. All operators different from these will never occur if they were not introduced with some unphysical - artificial, reasons.

The framework of functions over the phase space, that is often used for simultaneous representation of classical and quantum mechanics and investigation of the semiclassical limit, we find unsuitable for the related and well-known problems. We believe that unpleasant situations, the example of which is to end with possibly negative (quasi)distributions, are the consequence of the fact that this mathematical ambient is not wide enough to allow meaningful formulation of quantum mechanics.

The framework of commutative operators, on the other hand, certainly is not the minimal one needed for the representation of only classical mechanics. But, motivations for its introduction do exist. Some problems definitely ask for a framework wider than the usually used one. Only with the operatorial formulation of classical mechanics, the argumentation regarding the mentioned problem of measurement can be proceeded and completed without going in conflict with physical meaning, see Prvanović and Marić (2000).

The space $\mathcal{H}_q \otimes \mathcal{H}_p \otimes \mathcal{H}_r$ we find not only suitable, but also necessary for the meaningful representations of both mechanics and discussion of the semiclassical limit. So, with respect to this, it could be qualified as the minimal. However, one should be careful not to misuse its formal opportunities since, if one would do that, and only then, one would go out of physics. In short, the comparation of our approach with the alternative ones can be summarized in the following way: according to our opinion, it is better to be faced with the need to restrict the considerations and to do that *a priori* than to rectify it *a posteriori*.

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